Gauge-invariant resummation formalism and unitarity in non-commutative QED

Nicola Caporaso

Dipartimento di Fisica, Polo Scientifico Università di Firenze, INFN Sezione di Firenze
Via G. Sansone 1, 50019 Sesto Fiorentino, Italy
Email: caporaso@fi.infn.it

Sara Pasquetti

Dipartimento di Fisica, Università di Parma, INFN Gruppo Collegato di Parma
Parco Area delle Scienze 7/A, 43100 Parma, Italy
E-mail: pasquetti@fis.unipr.it

ABSTRACT: We re-examine the perturbative properties of four-dimensional non-commutative QED by extending the pinch techniques to the ϑ-deformed case. The explicit independence of the pinched gluon self-energy from gauge-fixing parameters and the absence of unphysical thresholds in the resummed propagators permits a complete check of the optical theorem for the off-shell two-point function. The known tachyonic dispersion relations are recovered within this framework, as well as their improved version in the (softly-broken) SUSY case. These applications should be considered as a first step in constructing gauge-invariant truncations of the Schwinger–Dyson equations in the non-commutative case. An interesting result of our formalism appears when considering the theory in two dimensions: we observe a finite gauge-invariant contribution to the photon mass because of a novel incarnation of IR/UV mixing, which survives the commutative limit when matter is present.

Contents

1. Introduction

The idea of introducing non-commutative space-time coordinates is not new [1] and has proved itself useful or interesting in a wide range of different fields. Theoretical high-energy physics observed a renewed interest toward non-commutativity in the last few years, due to its relation with string theory: the use of non-commutative geometry in this context was pioneered by Witten [2] in his formulation of open string field theory. More recently compactifications of M-theory on non-commutative tori were also studied in [3, 4]. Finally, after the discovery that spacelike non-commutativity emerges as an effective description of open strings in a constant NS-NS $B_{\mu\nu}$ field [5], this research line became really fashionable. String theory reduces in a particular low-energy limit to a quantum field theory on non-commutative Minkowski space-time characterized by the algebra

$$[\hat{x}_\mu, \hat{x}_\nu] = i\vartheta_{\mu\nu} : \tag{1.1}$$

this fact generated a flurry of activity (see [6] for reviews) to unveil the quantum properties of this novel class of models. Unfortunately the commutation relations (1.1) entail a breaking of Lorentz invariance, and difficulties arise at the quantum level in obtaining

the commutative $\vartheta \to 0$ limit in a sound way: These features are in open conflict with observations, making the phenomenological prospects of non-commutative models in particle physics quite thin [7 – 9].

Nevertheless non-commutative QFT is very interesting in its own right, presenting peculiar non-local interactions, non-perturbative solutions [10] and unconventional symmetries [11] that retain some properties of their string and D-brane ancestors. Concerning the quantum consistency of the theory, the loss of Lorentz invariance is not by itself a catastrophe. While important modifications to the dynamics, like non-trivial dispersion relations, may be introduced by breaking Lorentz symmetry, most of the fundamental aspects of relativistic QFT are retained, like microcausality, the CPT theorem, and so on [12]. Non-locality could instead drastically change the quantum dynamics. A non-commutative action can be constructed by deforming the ordinary, pointwise product of functions into the twisted convolution product ($\tilde{f}(k)$ is the Fourier transformed function)

$$f(x) \star g(x) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \, \tilde{f}(k) \, \tilde{g}(k - k') \, e^{-\frac{i}{2} \vartheta^{\mu\nu} k_\mu k'_\nu} e^{i \, k'_\rho x^\rho} \tag{1.2}$$

which manifestly induces terms with an arbitrarily high order of derivatives in the action. This in turn implies an odd (IR/UV) "mixing" of short and long distance scales by which, at the quantum level, ultraviolet divergences are transferred to the infrared domain [13]: this effect impairs the familiar Wilsonian point of view on renormalization [13 – 15]. Scalar theories have been widely studied and progresses have been recently reported in constructing a renormalizable perturbative expansion [16]. The case of a gauge theory is more difficult and no attempt has been done to prove systematically its consistency: a serious conceptual obstacle appears because non-commutative IR divergences induce tachyons at one-loop, and these destabilize the perturbative vacuum unless additional matter is introduced in a suitable way [17]. The relation between these tachyonic instabilities and string theory dynamics has been explored in [18].

Vacuum destabilization leaves one to ponder if a stable vacuum exists at all and, if this is the case, whether the breaking of Lorentz invariance might make it possible for the theory to develop exotic phases. These issues are intrinsically non-perturbative, and it would be natural to take advantage of the discretized formulation of non-commutative gauge theories [19 – 22]. Insisting on a continuum description, two approaches come instead to the mind: to write down an effective (CJT) action for composite operators [23], and making use of the Schwinger–Dyson equations (SD). Both approaches have been extensively exploited in the commutative framework along the years, and extensions to the non-commutative setup have been accomplished for the $\lambda \varphi^{\star 4}$ theory, suggesting the possibility for a transition toward "striped phases" [24]. The original proposal of the existence of a new vacuum state breaking of translational invariance has been further confirmed by analytical computations [25, 26] and by numerical simulations (in the lattice approach) [27]. Both the CJT effective action and the gap equations obtained from the SD equations encode a resummation of some infinite subset of Feynman diagrams. Unfortunately, in a gauge theory the mechanism by which the unphysical degrees of freedom cancel against each other calls into action a large number of different Feynman diagrams, so that a casual resummation of these

is almost certain to waste gauge-invariance, and yield a gauge-dependent answer to an ostensibly gauge-independent question. Already in the commutative case, for example, the SD make up a set of coupled non-linear equations and a truncation (either on the number of loops, or on the "order" of the n-point function examined, or both) is necessary to obtain a tractable gap equation. A "gauge-invariant resummable" formalism has been proposed [28] to achieve such a truncation without introducing gauge artifacts. These *pinch techniques* (PT) consist in composing the ghost and gauge-fixing-dependent degrees of freedom from different Feynman diagrams in such a way that gauge-independent propagators and vertices are defined *before* the Schwinger–Dyson equations are written down, making a gauge-independent truncation possible.

The pinch techniques are by now well established in the framework of ordinary gauge theories. They have been used to investigate the generation of an effective gluon mass in QCD [29], to properly describe resonant transition amplitudes and instable particles [31], [32] and in QFT at finite temperatures they are used to describe magnetic screening [33], to name just a few applications. Computations have been carried out explicitly to two-loops order [34] and in general gauges, covariant and not, showing the consistency and unicity of the definition of gauge-invariant propagators and vertices.

The main goal of this paper is the extension of the pinch techniques to the non-commutative setup; the possibility of writing down a gap equation is left for future work. A simpler application which we will discuss here is the analysis of the unitarity of the theory. For scalar field theories with purely spacelike non-commutativity no loss of unitarity appears, but its violation was observed in the timelike case in [35, 36] (that is, when $\vartheta^{\mu\nu}q_\nu$ has non-vanishing timelike components) in the guise of unphysical imaginary parts in the particle's self-energy. This issue was elucidated in [37], where the Dyson series was analysed employing a modified Hamiltonian density, and it was observed that the time ordering cannot be joined with the field contractions to yield the usual Feynman propagators with the noncommutative phase $e^{ik\vartheta q}$. Within this framework, unitarity can be read out in terms of graphs in [38]. Similar analyses have been attempted for non-commutative QCD [39, 40].

In the following, we shall concentrate on the case of purely spacelike noncommutativity. Unitarity of spacelike non-commutative gauge theories has only been checked for on-shell propagators and/or in specific gauges. In particular the authors of [40] have carried out the one-loop renormalization in a generic ξ-gauge: on-shell the dependence on ξ cancels together with all unphysical thresholds, but off-shell there is a host of unphysical thresholds depending on ξ. In view of the extension of the SD approach to non-commutative gauge theories, explicit gauge-independence of the off-shell propagator is certainly worth to be obtained. We will show that the pinch techniques can be extended to the non-commutative case, and give a solid check of unitarity for the off-shell propagators by computing the diagram that are connected to it by the optical theorem. Concerning the presence of the tachyonic pole and its cure by softly-breaking $\mathcal{N} = 4$ SUSY, we confirm the analysis presented in [12] and [14] where the computation was carried out within the background field method. This should not come as a surprise because the pinched propagator coincides with the background gauge field one for $\xi_Q = 1$ in the commutative setup.

New results are obtained when discussing the two-dimensional case. Gauge theories in two dimensions do not have propagating local degrees of freedom and this property should survive the non-commutative deformation: at the classical level, by simply choosing any axial gauge, the non-commutative $U(1)$ theory reduces to its commutative cousin, that is a trivial non-interacting theory. On the other hand, it is known that the relation between perturbative and non-perturbative aspects is subtle for $2D$ gauge theories [41, 42] and non-commutativity has already produced some surprises when computing Wilson loops [43 – 46]. In the present case, we would expect naively no correction to the free-propagator when matter is absent, even in the non-commutative case, due to the gauge-invariant meaning of the pinched self-energy. We obtain instead a surprising result: working in a covariant gauge, the dimensionally regularized theory, in the limit $D \to 2$, exhibits, even in absence of matter, a non-trivial ϑ-dependent correction to the dispersion relation which owes its finiteness to a fine cancellation between planar and non-planar contribution. Moreover as $\vartheta \to 0$, we observe an anomalous behavior from the matter contributions, apparently inducing a mass for the photon: this is analogous to what happens in three-dimensions for the Chern-Simons term generated by Majorana fermions [47]. On the other hand, when $\vartheta \to \infty$ the original infrared divergences, tamed by non-commutativity, reappear, leaving us with a ultraviolet logarithmic divergent term. This is a twisted incarnation of the UV-IR effect. We remark that our result should be meaningful, due to the use of the gauge-invariant pinched self-energy.

The plan of the paper is the following: in section 2 we describe how to extend the pinch techniques to the non-commutative case and we present the computation of the "pinched" gluon self-energy. In section 3 we check the on-shell behavior of the pinched propagator, we study unitarity in the spacelike case by proving the optical theorem and discuss analyticity by analyzing the dispersion relation[1] connecting the real and imaginary parts of the gluon self-energy. Section 4 is devoted to study the two-dimensional gauge theory, with and without matter. Section 5 contains some concluding remarks and appendices A and B are devoted to some technical aspects of the computations.

2. Pinch techniques

One of the main problems in quantizing gauge theories is to deal with the unphysical poles and thresholds that generically plague local Green functions. On a theoretical ground this is not an issue: the well-known answer is to limit the analysis to gauge-invariant quantities, which are free of such troubles. On a practical ground, however, the question remains. In fact, simple invariant quantities like the S-matrix elements are only defined on-shell, and the off-shell physics is out of their grasp. Besides, more general off-shell observables like

---

[1]We warn the reader about the two meanings of the term "dispersion relation" which are used here and in the literature. On the one hand, it can be taken to mean the dependency of a particle's energy on its momentum, $E^2(\vec{p})$, which becomes non-trivial because of the breaking of Lorentz invariance. The second use refers to the relation between the real and imaginary parts of an analytic function. The distinction is always clear from the context.

the Wilson and Polyakov loops are non-local and this makes them much more difficult to compute.

Pinch techniques (PT), in the original formulation by Cornwall [28, 29], provide a manageable solution to this problems. These consist in an algorithm that rearranges the S-matrix elements of gauge theories and produces off-shell proper correlation functions which satisfy the same Ward identities (WI) as those produced by the classical Lagrangian [29, 30]. The PT off-shell Green functions, in addition to being gauge invariant by construction, also satisfy basic theoretical requirements such as unitarity, analiticity and renormalizability. They can be also used as the building blocks of gauge-invariant Schwinger–Dyson equations, which allow to discuss non perturbative questions as vacuum stability, dynamical mass generation [29], and the behavior of unstable states [32] in the commutative setup. A comprehensive discussion of this topic is out of the goals of the present paper (see [48] for excellent reviews); here, instead, we shall briefly outline how these techniques have been applied in commutative theories to the case we are interested in: the one-loop vacuum polarization.

In a nutshell, PT consists in a judicious use of the cancellations that underlie the well-known gauge invariance of the S-matrix. One can concentrate on a two-particle process, and identify which scattering amplitude contains the relevant information about self-energy by simply looking at the structure of the exchanged momenta. The classical choice is the fermion-antifermion scattering process $f(p_1) + \bar{f}(p_2) \to f'(k_1) + \bar{f}'(k_2)$, but any other two-particle process would be adequate, since the result does not depend on this choice [49]. It is convenient to paramterize the one-loop amplitude $f\bar{f} \to f'\bar{f}'$ in any covariant ξ-gauge[2] so that the structure of the exchanged momenta is most evident:

$$
\begin{aligned}
\langle f(p_1)\bar{f}(p_2)|T(s,t)|f'(k_1)\bar{f}'(k_2)\rangle &= \Gamma^\mu(p_1,p_2)\Delta^{(\xi)}_{\mu\alpha}(s)\Pi^{(\xi)\alpha\beta}(s)\Delta^{(\xi)}_{\beta\nu}(s)\Gamma^\nu(k_1,k_2) + \\
&+ \Gamma_1^{(\xi)\mu}(p_1,p_2)\Delta^{(\xi)}_{\mu\nu}(s)\Gamma^\nu(k_1,k_2) + \\
&+ \Gamma^\mu(p_1,p_2)\Delta^{(\xi)}_{\mu\nu}(s)\Gamma_1^{(\xi)\nu}(k_1,k_2) + B^{(\xi)}(p_1,p_2,-k_1,-k_2),
\end{aligned}
\tag{2.1}
$$

where s and t are Mandelstam variables. In (2.1) the symbol $\Pi^{(\xi)\alpha\beta}(s)$ designates the gauge-dependent vacuum polarization, and $\Gamma_1^{(\xi)\nu}$ and $B^{(\xi)}$ denote respectively the one-loop correction to the cubic $\bar{f}Af$ vertex and box graphs; $\Delta^{(\xi)}_{\mu\nu}(s)$ and Γ^μ stand for the tree-level gluon propagator and vertex. External current are included in vertices $\Gamma^{(\xi)\nu}$ and $\Gamma_1^{(\xi)\nu}$. Finally, the superscript (ξ) over each term in (2.1) marks the intrinsic gauge dependence of the different contributions. The above decomposition corresponds to summing the diagrams depicted in figure 1.

Gauge-invariance of the S-matrix ensures that the sum of all the graphs $(a) + (b) + (\bar{b}) + (c) + (\bar{c})$ of figure 1 is independent of the gauge parameter ξ, and so the matrix element $\langle f(p_1)\bar{f}(p_2)|T(s,t)|f'(k_1)\bar{f}'(k_2)\rangle$ must also be independent of ξ. The cancellation occurring between these different diagrams is well-known but rather intricate, and it has some surprises in store. By examining the analytical structure of this scattering amplitude one can easily identify a few different sectors: some terms depend solely on s, and there are

---

[2]With this expression, we mean the usual gauge breaking term given by $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$.
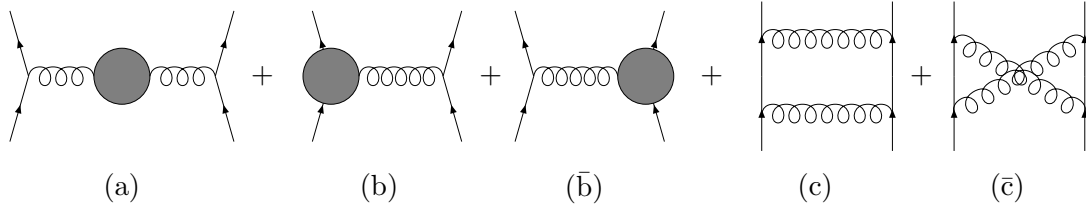
Figure 1: The contribution (a) corresponds to $\Pi^{(\xi)\alpha\beta}$; (b) and $(\bar{b})$ contain the one-loop correction $\Gamma^{(\xi)\mu}$ to the vertex, and $(c),(\bar{c})$ are the box diagrams $B^{(\xi)}$.

other, more complicated terms that carry an intrinsic dependence both on s and t. This suggests that the cancellations responsible for the invariance of (2.1) are not "global", but occur separately in different channels, so that more than one invariant structure is buried there.

That this naive observation leads to a concrete and useful application is non-trivial. Cornwall [29] has indeed shown that it is possible to rearrange (2.1) in the following form

$$
\begin{aligned}
\langle f(p_1)\bar{f}(p_2)|T(s,t)|f(k_1)\bar{f}(k_2)\rangle = {} & \Gamma^\mu(p_1,p_2)\Delta^{(\xi)}_{\mu\alpha}(s)\widehat{\Pi}^{\alpha\beta}(s)\Delta^{(\xi)}_{\beta\nu}(s)\Gamma^\nu(p_3,p_4) + \\
& + \widehat{\Gamma}^\mu(p_1,p_2)\Delta^{(\xi)}_{\mu\nu}(s)\Gamma^\nu(k_1,k_2) + \\
& + \Gamma^\mu(p_1,p_2)\Delta^{(\xi)}_{\mu\nu}(s)\widehat{\Gamma}^\nu(k_1,k_2) + \widehat{B}(p_1,p_2,-k_1,-k_2),
\end{aligned}
\tag{2.2}
$$

where the *pinched* polarization tensor $\widehat{\Pi}^g_{\alpha\beta}(s)$, vertex $\widehat{\Gamma}^\mu$ and box $\widehat{B}$ are separately gauge-invariant and independent of ξ. The surviving dependence on ξ in $\Delta^{(\xi)}_{\mu\nu}(s)$ is irrelevant, and it drops out as soon as the tree level propagator hits the external current in Γ^μ or $\hat{\Gamma}^\mu$; it has been left just for future convenience.

Practically, the pinched representation (2.2) is obtained in two steps. First one extracts from the second and third lines of (2.1) those contributions which depend solely on the Mandelstam variable s. Then one combines these contributions with $\Pi^{(\xi)\alpha\beta}$ to yield a gauge-invariant quantity $\widehat{\Pi}^g_{\mu\nu}$. Clearly, some more work is required in the second and the third line to single out $\widehat{\Gamma}$ and $\widehat{B}$, but as long as one is interested only in the pinched vacuum polarization, this step is not relevant. More details can be found in the literature.

The experienced reader may doubt that the procedure we just sketched out is uniquely determined. In fact, since we are dealing with scattering amplitudes, any redefinition that is proportional to the equations of motion leaves equation (2.2) unaltered. To fix this ambiguity and consistently promote off-shell the pinched Green functions one imposes, apart from a ξ-independence, a few reasonable constraints:

1. The resulting Green's functions must be free from unphysical poles and thresholds.

2. The Green functions must satisfy the tree-level Ward identities dictated by the classical Lagrangian.

3. The Green functions must be resummable and compatible with the off-shell Schwinger–Dyson equations. This means, for example, that the ξ-dependence must cancel before integrating over loop momenta.

4. The resummed Green functions must revert, when evaluated on-shell, to the conventional ones. This means that the resummation prescription must leave the position of the poles unchanged, since this is a gauge-invariant information.

These constraints make the pinched Green's functions uniquely defined.

In the commutative setup there would be additional items in this list, making reference to the constraints dictated by unitarity and analyticity. Since we deal with non-commutative field theories, where unitarity might be jeopardized by nonlocal effects, we shall drop these requirements and proceed without imposing them. As we will show, the outcome of this approach is threefold: first of all, it shows the applicability of the pinch techniques in the non-commutative framework, even though the additional constraints of unitarity and analyticity are not imposed. Secondly, it provides us with a gauge-invariant test of unitarity for the non-commutative theory; and finally, it allows for an investigation of the analyticity properties through the analysis of the dispersion relation.

We are now ready to illustrate some details of the computations leading to the pinched non-commutative vacuum polarization. As is well-known in the literature, the tensorial structures involved in the one-loop *un*integrated amplitudes are unchanged by non-commutativity, even though Lorentz invariance is broken[3]. The only difference consists in the presence of a trigonometric factor inside the vertices that spreads the nonlocal information, and depends on the loop and external momenta (see appendix A for the Feynman rules). This new ingredient, however, does not impair the pinching procedure, and it can be examined separately as we now show. Consider, for example, the first diagram in figure 2. Employing the Feynman rules of appendix A for the gauge fields and Dirac fermions, the trigonometric factors associated with this amplitude are

$$e^{ip_1\vartheta p_2/2 - i(k_2-k_1)\vartheta r/2}\sin\left(\frac{q\vartheta\ell_2}{2}\right) = e^{ip_1\vartheta p_2/2 - i(k_2-k_1)\vartheta(k_2-\ell_2)/2}\sin\left(\frac{q\vartheta\ell_2}{2}\right) =$$
$$= e^{ip_1\vartheta p_2/2 + ik_1\vartheta k_2/2}\left[e^{-iq\vartheta\ell_2}\sin\left(\frac{q\vartheta\ell_2}{2}\right)\right]. \quad (2.3)$$

This contains an overall factor associated with the scattering of two fermions, which depends only on the external momenta, and another quantity, in brackets, which is relevant for the loop integration. However when taking into account the contribution of the "mirror" diagram (the second one in figure 2), one sees that there the dependence on the phase factors is got by simply setting $q \to -q$ in (2.3). Summing the pinch contributions of this two diagrams, the trigonometric factors recombine into

$$2ie^{ip_1\vartheta p_2/2 + ik_1\vartheta k_2/2}\left[\sin^2\left(\frac{q\vartheta\ell_2}{2}\right)\right]. \quad (2.4)$$

The quantity in brackets has the right structure to mimic the phase factor of the vacuum polarization, so it can be recombined with it to yield a pinched polarization tensor. The process we just outlined repeats itself unaltered for all the other diagrams. Thus, in the

---

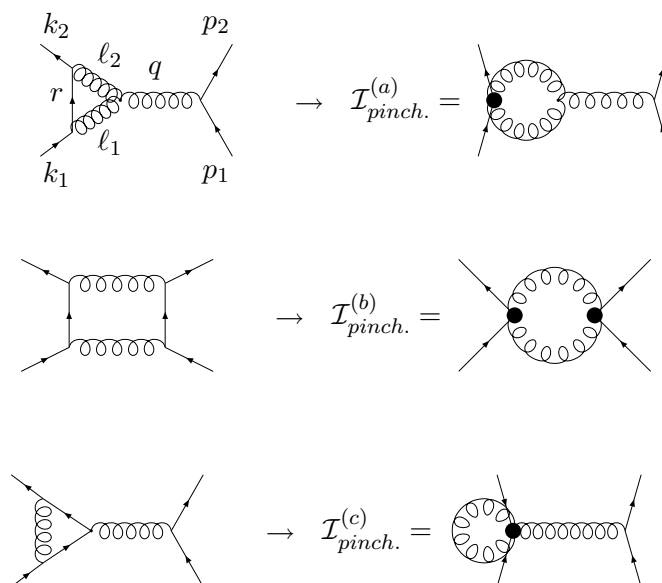[3]A new tensorial structure will emerge only after integrating over loop momentum.

Figure 2: One-loop propagator-like contributions to the pinch technique.

following, we shall focus just on the tensorial structure, and we shall reinsert the necessary trigonometric factors upon integrating.

The next step consists in extracting the s-dependent contributions from diagrams (a), (b) and (c) as shown in figure 2. This is accomplished by means of a classical trick. Let us denote respectively with r, ℓ_1 and ℓ_2 the momenta of fermion and of the two gluons running in the loop in the first diagram in figure 2. In addition, call the external momenta attached to this loop k_1 and k_2. Then the following silly Ward identities hold

$$\ell_1 = (\not{r} - \not{k}_1) = (\not{r} - m) - (\not{k}_1 - m) = S^{-1}(r) - S^{-1}(k_1)$$
$$\ell_2 = (\not{k}_2 - \not{r}) = (\not{k}_2 - m) - (\not{r} - m) = S^{-1}(k_2) - S^{-1}(r), \tag{2.5}$$

where S is the free fermion propagator. Equations (2.5) state that any term of the form ℓ_1 or ℓ_2 present in the diagrams spawns two terms: one is proportional to the equation of motion of the external leg, and it can be dropped; the second one is proportional to the inverse of the propagators of the fermion running inside the loop. The net effect of this procedure is to squeeze away "pinch" the internal fermionic propagators. Graphically, this mechanism is represented by the diagrams appearing on the r.h.s. of figure 2. We have obtained effective Feynman diagrams where one or both of the fermion propagators have been pinched: these diagrams exhibit clearly their dependence on merely s. The same trick allows us to handle the other two diagrams in figure 2. In the present discussion we have neglected diagrams governing the renormalization of the external legs, since their total effect on the pinching procedure vanishes.

Now we collect the different contributions to the pinched vacuum polarization. We start from those coming from the diagram (a) in figure 1. These include the three ordinary
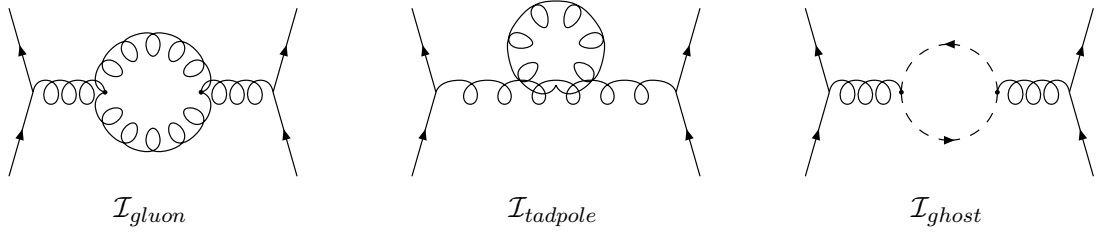
Figure 3: Ordinary contributions to $\Pi_{\alpha\beta}^{(\xi)}$.

contributions to the vacuum polarization tensor of non-commutative QED, represented in figure 3. Their value is reported below:

$$\mathcal{I}_{gluon} = \left[\frac{g_{\alpha\beta}}{k^2} + \frac{2\,q^2\,g_{\alpha\beta} + (2D-3)k_\alpha k_\beta}{k^2 p^2} + (1-\xi)\left(\frac{-k^2 g_{\alpha\beta} + k_\alpha k_\beta}{p^4} - 2\frac{q^2 k_\alpha k_\beta}{k^2 p^4} + \right. \right.$$
$$\left. \left. + 2\frac{q^2 g_{\alpha\beta}}{k^4} - \frac{q^4 g_{\alpha\beta}}{k^2 p^4} \right) + (1-\xi)^2 \left(\frac{q^4 k_\alpha k_\beta}{k^4 p^4} \right) \right] + (k \leftrightarrow p), \tag{2.6}$$

$$\mathcal{I}_{ghost} = -\left(\frac{k_\alpha p_\beta + k_\beta p_\alpha}{k^2 p^2} \right) \qquad \mathcal{I}_{tadpole} = 2(1-D)\frac{g_{\alpha\beta}}{k^2} - 2(1-\xi)\left(\frac{g_{\alpha\beta}}{k^2} - \frac{k_\alpha k_\beta}{k^4} \right). \tag{2.7}$$

We have denoted with q the external momentum and with k, p the loop momenta. Then we have pinched contributions from the s-parts of diagrams (b) and (c), as shown in figure 2.

$$\mathcal{I}_{pinch.}^{(a)} = 4q^2 \frac{g_{\alpha\beta}}{k^2 p^2} + 2(1-\xi)\left[\frac{q^2\,k_\beta\,k_\alpha}{k^4\,p^2} + \frac{q^2\,p_\alpha\,p_\beta}{k^2\,p^4} + \left(\frac{q^4}{k^2\,p^4} + \frac{q^4}{k^4\,p^2} \right)g_{\alpha\beta} - \right.$$
$$\left. - \left(\frac{1}{k^4} + \frac{1}{p^4} \right)q^2 g_{\alpha\beta} \right] - 2(1-\xi)^2 \left(\frac{q^4\,p_\alpha\,p_\beta}{k^4\,p^4} \right), \tag{2.8}$$

$$\mathcal{I}_{pinch.}^{(b)} = 2(1-\xi)\left(-\frac{q^4 g_{\alpha\beta}}{k^2 p^4} - \frac{q^4 g_{\alpha\beta}}{k^4 p^2} \right) + 2(1-\xi)^2 \frac{q^4 p_\alpha p_\beta}{k^4 p^4}, \tag{2.9}$$

$$\mathcal{I}_{pinch.}^{(c)} = -2(1-\xi)\left(\frac{q^2 g_{\alpha\beta}}{k^4} + \frac{q^2 g_{\alpha\beta}}{p^4} \right). \tag{2.10}$$

We remark that these are all independent of the dimension D. Summing up all these contributions we find that the the pinched polariszation tensor is

$$\widehat{\Pi}_{\alpha\beta} = -2g^2 \int \frac{\mathrm{d}^D k}{(4\pi)^{D/2}} \frac{8(q^2 g_{\alpha\beta} - q_\alpha q_\beta) + (4-2D)(k^2 g_{\alpha\beta} - k_\alpha k_\beta)}{k^2(k+q)^2} \sin^2\left(\frac{q\vartheta k}{2} \right), \tag{2.11}$$

where we have restored the relevant trigonometric factor. Equation (2.11) is particularly intriguing not only because it is manifestly transverse, but also because it signals the possible presence of *evanescent terms* in two dimensions: there is a potential competition between the second integral, which is logarithmically ultraviolet divergent, and its vanishing coefficient when approaching $D = 2$. This issue will be discussed in detail in Sect.4.

The computations leading to the Euclidean version of the integrals in (2.11) are shown in appendix B. The final result takes on the form

$$\widehat{\Pi}^g_{\mu\nu} = \widehat{\Pi}_c(q^2, |\tilde{q}|^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{|\tilde{q}|^2}\right) + \left(\widehat{\Pi}_\vartheta(q^2, |\tilde{q}|^2) + \widehat{\Pi}_c(q^2, |\tilde{q}|^2)\right)\frac{\tilde{q}_\mu \tilde{q}_\nu}{|\tilde{q}|^2}. \quad (2.12)$$

Where $\tilde{q}^\nu = \vartheta^{\mu\nu}q_\nu$ and the conventional decomposition of $|\tilde{q}|^2$ is $|\tilde{q}|^2 = q \bullet q = \Theta_E(p_0^2 + p_1^2) + \Theta_B(p_2^2 + p_3^2)$, and (as remarked above) we are considering the case $\Theta_E = 0$. The notation here is chosen to underline the existence of two spacetime invariants for non-commutative gauge theories, q^2 and $|\tilde{q}|^2$. In the following, to lighten the notation, the dependence on the two invariants is understood. The explicit values of the two functions are given by the following integrals over the Feynman parameters

$$\widehat{\Pi}_c = -\frac{g^2}{(4\pi)^{D/2}}\int_0^1 \mathrm{d}x \frac{8q^2 + (4-2D)(-M^2 + q^2x^2)}{(M^2)^{2-D/2}}$$
$$\times \left(\Gamma(2 - \frac{D}{2}) - 2\left(\frac{|\tilde{q}|M}{2}\right)^{2-\frac{D}{2}} K_{2-\frac{D}{2}}(M|\tilde{q}|)\right),$$
$$\widehat{\Pi}_\vartheta = g^2 \frac{(4-2D)}{(4\pi)^{D/2}} 2 \int_0^1 \mathrm{d}x\, 2M^2 \left(\frac{|\tilde{q}|}{2M}\right)^{2-D/2} K_{-D/2}(M|\tilde{q}|), \quad (2.13)$$

where $M \equiv \sqrt{x(1-x)q^2}$ and K_n is the modified Bessel function of the second kind.

3. The four-dimensional theory

We remarked above that in the commutative case one further requires, among the defining properties of pinch-technique resummed amplitudes, the off-shell optical relations, analyticity and invariance of the position of the poles. In the non-commutative case the situation is more involved because such properties could be spoiled by non-commutative effects. This section is devoted to a detailed analysis of these issues. We begin by studying the on-shell properties of the pinched propagator in the supersymmetric extension of the theory: as we discussed above it is an important consistency check that the on-shell physics is untouched by the pinching procedure. In the following sections we discuss the unitarity and analyticity properties of the pinched propagator, reverting to the matterless case.

3.1 Dispersion relations in the supersymmetric extension of the theory

It is well known that in four dimensions the UV/IR mixing induces a quadratic IR divergence in the self-energy, which produces a tachyonic divergence in the dispersion relation. As observed in [12],[14], one can recover vacuum stability by introducing a sufficient amount of supersymmetric matter: SUSY in fact, by improving the UV behavior of quantum loops, acts as a regulator for the infrared divergences of a non-commutative theory, because the UV effects are tamed before they can couple to $\vartheta_{\mu\nu}$ and induce large-distance divergences. Adding the contribution of n_f fermions and n_s scalars we find

$$\widehat{\Pi}_c = \frac{g^2}{(4\pi)^{D/2}}\int_0^1 \mathrm{d}x \left[-\frac{[2(2-D)(-M_g^2 + q^2x^2) + 8q^2]}{(M_g^2)^{2-D/2}}A(m_g) + \right.$$

$$+ \frac{2 \, D \, q^2 x(1-x)}{(M_f^2)^{2-D/2}} \sum_f A(m_f) + \frac{q^2(4x^2-1)}{(M_s^2)^{2-D/2}} \sum_s A(m_s) \bigg], \tag{3.1}$$

$$\widehat{\Pi}_\vartheta = \frac{g^2}{(4\pi)^{D/2}} \int_0^1 \mathrm{d}x \, \bigg[2(4-2D)B(m_g) + 2D \sum_f B(m_f) - 4 \sum_s B(m_s) \bigg]. \tag{3.2}$$

Here we have included softly supersymmetry-breaking masses m_f, m_s, m_g for fermions, scalars and gluons. The dependence on the masses is contained in the functions M_i:

$$A(m_i) = \left[\Gamma(2-\frac{D}{2}) - 2 \left(\frac{|\tilde{q}|M_i}{2} \right)^{2-\frac{D}{2}} K_{2-\frac{D}{2}}(M_i|\tilde{q}|) \right], \tag{3.3}$$

$$B(m_i) = 2M_i^2 \left(\frac{|\tilde{q}|}{2M_i} \right)^{2-D/2} K_{-D/2}(M_i|\tilde{q}|), \tag{3.4}$$

where $M_i(m_i,q) = m_i^2 + x(1-x)q^2$. The planar part of the vacuum polarization is UV-divergent for $D \to 4$ and needs to be renormalized. We will perform this procedure just like in the commutative case: we choose a subtraction scale μ, so that

$$A(m_i) \to - \left[\log \left(\frac{M_i}{\mu} \right) + K_0(M_i|\tilde{q}|) \right]. \tag{3.5}$$

Two limits of these functions are worthy of being considered.

One of the virtues of the pinch techniques is their ability to compute the β-functions, and we would like to recover these in the appropriate limit. This limit consists in reaching the ultraviolet by taking the arguments of the Bessel functions to be large, so that their contribution is exponentially suppressed, together with $q \gg m_i$:

$$\widehat{\Pi}_c = \frac{1}{4\pi^2} \left(\frac{11}{3} - \frac{2}{3}n_f - \frac{1}{6}n_s \right) \log \left(\frac{q}{\mu} \right). \tag{3.6}$$

We recover the standard β-function for softly broken susy gauge theories: it is a further check of the validity of the PT resummation prescription.

Of course one could compute the β-function using the *background field method* (BFM); however, as pointed out in [32], BFM n-point functions display a residual dependence on the gauge parameter ξ_Q employed in fixing the gauge for the quantum fields inside the loops, and this may lead to unphysical thresholds. Requiring the absence of such unphysical effects forces the choice $\xi_Q = 1$, which cannot be otherwise motivated; in this case the one-loop n-point functions evaluated in BFM and the ones we computed using the pinch techniques coincide.

A second limit concerns approaching the infrared with small arguments of the Bessels, so that:

$$A(m_i) \to \log \left(|\tilde{q}|\mu \right). \tag{3.7}$$

In this limit we have:

$$\widehat{\Pi}_c = -\frac{1}{4\pi^2} \left(\frac{11}{3} - \frac{2}{3}n_f - \frac{1}{6}n_s \right) \log \left(|\tilde{q}|\mu \right). \tag{3.8}$$
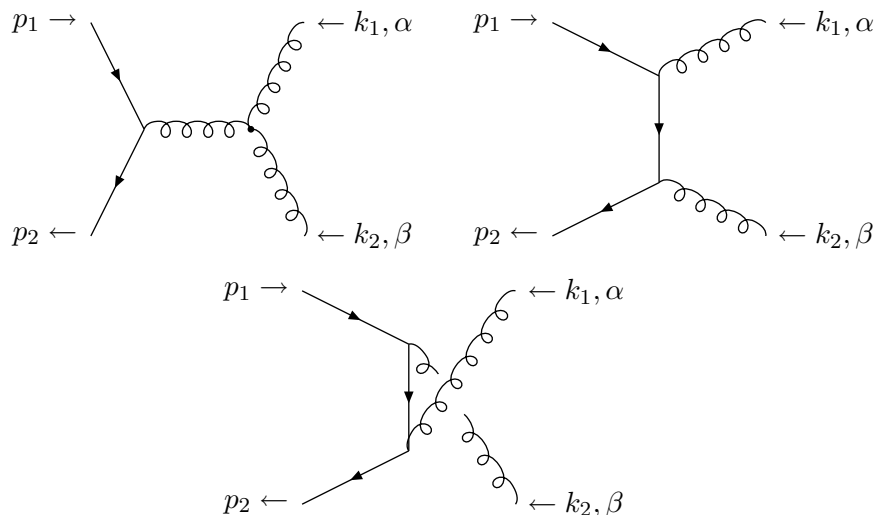
Figure 4: *s*- *t*- *u*-channel amplitudes

As already pointed out in [12], this expression shows that the running of the coupling constant in the infrared is similar to the one in the ultraviolet. A different sign indicates that the theory becomes weakly coupled at low energy. The duality $q \to \frac{1}{|\tilde{q}|}$ is thus interpreted as another incarnation of the UV-IR mixing. The expression we found for the self-energy, through equations (3.1) and (3.2), coincides with the one found in [12] and [14]. In particular, the pure gluon contribution to the equation for the position of the poles gives rise to the well-known tachyonic dispersion relation. This last feature implies that the PT-resummed amplitudes reduce to the unpinched value when evaluated on-shell as they should. In other words, the resummation prescription does not modify the position of the poles.

3.2 Optical theorem and unitarity

Having verified the on-shell properties of the pinched propagator we move to the analysis of the off-shell physics. Let us first of all show how the pinch techniques can be employed to check the optical theorem for *off-shell* two-point functions. For on-shell matrix elements the optical theorem states that if the S-matrix is unitary then:

$$\Im\mathrm{m}\langle q\bar{q}|T|q\bar{q}\rangle = \frac{1}{2}\left(\frac{1}{2}\right)\int d\Omega \langle q\bar{q}|T|gg\rangle\langle gg|T|q\bar{q}\rangle. \tag{3.9}$$

On the left hand side we have the S-matrix element for a q-$\bar{q}$ scattering process (the one from which the pinched vacuum polarization is obtained), and on the right hand side we have the amplitudes for quark-gluon scattering. The diagrams contributing to this amplitude are obtained by cutting through the S-matrix and are displayed in figure 4. The factor of 1/2 appears because the final on-shell gluons are identical particles.

In the last section we built a gauge-independent self-energy by resumming all the one-loop *s*-channel contributions to the matrix elements on the left hand side. The analogue of this procedure on the right-hand side consists in recasting it as a sum of gauge-independent

s, t and u-channel contributions. If the two sides match, we will obtain a direct check of unitarity in the s-channel. Let then $\mathcal{M}$ be the $q\bar{q}$ scattering S-matrix element, and $\mathcal{T}$ be the q-g scattering amplitude:

$$\mathcal{M} \doteq \langle q\bar{q}|T|q\bar{q}\rangle \ , \ \mathcal{T} \doteq \langle q\bar{q}|T|gg\rangle. \tag{3.10}$$

$\mathcal{T}$ consists of the $\mathcal{T}_s$, $\mathcal{T}_t$ and $\mathcal{T}_u$ contributions displayed in figure 4: we must take the squared modulus and sum over all physical gluon polarizations.

The matching of different channels from $\mathcal{M}$ and $\mathcal{T}$ is non-trivial, we shall show, in fact, that the "pinched optical theorem" relates $\mathcal{M}_s$ to the whole of $\mathcal{T}_s$ plus pieces from $\mathcal{T}_t$ and $\mathcal{T}_u$. On second thoughts this comes as no surprise: by cutting through the box diagram one obtains $\mathcal{T}_t$ and $\mathcal{T}_u$, and it is just to be expected that if the box gives pinch contributions, so must $\mathcal{T}_t$ and $\mathcal{T}_u$. In the following we specialize to $D = 4$ and adopt Minkowskian signature. In order to analytically continue the results of the previous section one needs to send:

$$p_E^2 \to -p_M^2, \qquad \tilde{p}_E^2 \to -\tilde{p}_M^2,$$
$$(p \bullet p)_E = \Theta_B(p_2^2 + p_3^2) \to (p \bullet p)_M = \Theta_B(p_2^2 + p_3^2).$$

Our computation follows closely [32]. The contributions are:

$$\mathcal{T}_s^{\alpha\beta} = J^\mu(p_1, p_2) \sin\frac{-p_2 \vartheta p_1}{2} \Delta_{\mu\nu}(q) V^{\nu\alpha\beta}(q, k_1, k_2)$$
$$\mathcal{T}_{t+u}^{\alpha\beta} = J^{\alpha\beta}(p_1, p_1 + k_1, p_2) \sin\frac{p_2 \vartheta k_2}{2} \sin\frac{p_1 \vartheta k_1}{2} + (k_1, \alpha \leftrightarrow k_2, \beta), \tag{3.11}$$

where $J^\mu(p_1, p_2) = g\bar{v}(p_2)\gamma^\mu u(p_1)$ and $J^{\alpha\beta}(p_1, p_1 + k_1, p_2) = g^2 \bar{v}(p_2)\gamma^\alpha S_F(p_1 + k_1)\gamma^\beta u(p_1)$, while $V_{\nu\alpha\beta}(q, k_1, k_2)$ is the three-gluon vertex. The optical theorem states that

$$\Im\mathrm{m}\{\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u\} = \int \frac{d\Gamma}{4} \left(\mathcal{T}_s^{\mu\nu} + \mathcal{T}_{t+u}^{\mu\nu}\right) P_{\mu\rho}(k_1) P_{\nu\sigma}(k_2) \left(\mathcal{T}_s^{*\rho\sigma} + \mathcal{T}_{t+u}^{*\rho\sigma}\right),$$

where $\mathcal{M}_s$ contains the pinched vacuum polarization tensor $\widehat{\Pi}_{\mu\nu}$,

$$\Im\mathrm{m}\,\mathcal{M}_s = \int d\Gamma \frac{1}{4q^4} J^\mu(p_1, p_2)\widehat{\Pi}_{\mu\nu} J^{*\nu}(p_1, p_2) \sin^2\frac{k_1 \vartheta k_2}{2} \sin^2\frac{p_1 \vartheta p_2}{2} \tag{3.12}$$

and $P_{\mu\nu}$ represents the polarization tensor (the sum over the gluons' physical polarizations)

$$P_{\mu\nu}(q, \eta) \doteq -g_{\mu\nu} + \frac{\eta_\mu q_\nu + q_\mu \eta_\nu}{(\eta \cdot q)} + \frac{\eta^2 q_\mu q_\nu}{(\eta \cdot q)^2}. \tag{3.13}$$

Before we plunge into the calculations, it is important to make a few remarks concerning the consistence of the pinch techniques as they are applied to the squared modulus of $\mathcal{T}$. One should notice that in this case we have a dependence on two "gauge-parameters": the gauge-fixing one, ξ, and η_μ, which is introduced upon choosing the two independent physical polarization vectors to be summed over. We demand that the pinch contributions from the second and third graphs cancel the ξ and η_μ-dependence of the first diagram. Let

us begin by showing the independence from ξ. By decomposing the three-gluon vertex as $V^T = V^F + V^P$ we have

$$V^F_{\mu\nu\rho}(q,p,k) = g\left[(p-k)_\mu g_{\nu\rho} + 2q_\nu g_{\mu\rho} - 2q_\rho g_{\mu\nu}\right]\sin\frac{p\vartheta k}{2}, \qquad (3.14)$$

$$V^P_{\mu\nu\rho}(q,p,k) = g[k_\rho g_{\mu\nu} - p_\nu g_{\mu\rho}]\sin\frac{p\vartheta k}{2}. \qquad (3.15)$$

We observe that for on-shell gluons, which obey $k^\mu P_{\mu\nu} = 0$ and $k^2 = 0$, the term $V^P_{\mu\nu\rho}$ dies upon hitting $P_{\mu\rho}\, P_{\nu\sigma}$. The s-channel's explicit dependence on the ξ-gauge disappears thanks to the fact that the gauge-fixing term is longitudinal, so it also drops off as it hits the conserved external fermionic current[4]. Then the s-channel propagator boils down to its Feynman-gauge values and all dependence on ξ vanishes. We must still prove that also the dependence on η_μ gets cancelled as well.

Luckily, all the phases factorize exactly like in the first section and the following set of Ward identities can be obtained:

$$k_1^\alpha (\mathcal{T}_s)_{\alpha\beta} = \left(2\frac{k_1^\mu k_2^\beta}{q^2} - g^{\mu\beta}\right) J^\mu(p_1,p_2)\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2},$$

$$k_2^\beta (\mathcal{T}_s)_{\alpha\beta} = \left(2\frac{k_2^\mu k_1^\alpha}{q^2} + g^{\mu\alpha}\right) J^\mu(p_1,p_2)\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2},$$

$$k_1^\alpha (\mathcal{T}_{t+u})_{\alpha\beta} = J_\beta(p_1,p_2)\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2},$$

$$k_2^\beta (\mathcal{T}_{t+u})_{\alpha\beta} = -J_\alpha(p_1,p_2)\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2},$$

$$k_1^\alpha k_2^\beta (\mathcal{T}_s)_{\alpha\beta} = -k_{2\,\mu}J^\mu(p_1,p_2)\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2},$$

$$k_1^\alpha k_2^\beta (\mathcal{T}_{t+u})_{\alpha\beta} = k_2^\beta J_\beta(p_1,p_2)\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2}.$$

Defining:

$$\mathcal{G} \doteq -\frac{k_2^\mu}{q^2}J_\mu(p_{\text{in}},p_{\text{out}})\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2} = \frac{k_1^\mu}{q^2}J_\mu(p_{\text{in}},p_{\text{out}})\sin\tfrac{p_1\vartheta p_2}{2}\sin\tfrac{k_1\vartheta k_2}{2}, \quad (3.16)$$

we arrive at the relevant cancellation laws displayed in figure 5

$$k_1^\alpha (\mathcal{T}_{\text{tot}})_{\alpha\beta} = 2k_2^\beta \mathcal{G}, \qquad (3.17)$$

$$k_2^\beta (\mathcal{T}_{\text{tot}})_{\alpha\beta} = 2k_1^\alpha \mathcal{G}, \qquad (3.18)$$

$$k_1^\alpha k_2^\beta (\mathcal{T}_{\text{tot}})_{\alpha\beta} = 0. \qquad (3.19)$$

Thanks to this cancellation it is straightforward to recast the optical theorem as

$$\Im m\{\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u\} = \frac{1}{4}\int d\Gamma\Bigg[\underbrace{(\mathcal{T}_s^{\mu\nu}\mathcal{T}^*_{s\,\mu\nu} - 8\mathcal{G}\mathcal{G}^*)}_{\text{propagator-like}} + \underbrace{(\mathcal{T}_s^{\mu\nu}\mathcal{T}^{*\,\mu\nu}_{t+u} + \mathcal{T}_{t+u}^{\mu\nu}\mathcal{T}^*_{s\,\mu\nu})}_{\text{vertex-like}} +$$

---

[4]Had we not used a covariant gauge, current conservation would not have been sufficient to guarantee the gauge fixing independence. Like in the commutative case [32], in this case one resorts to the following Ward identity

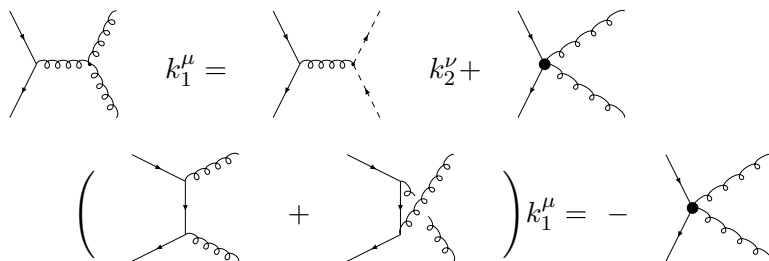$$q^\mu V^F_{\mu\nu\rho}(q,-p-k) = (p^2 - k^2)g_{\nu\rho}.$$

Figure 5: s- channel (first line), and t- and u- channels cancellations

$$+ \underbrace{\mathcal{T}_{t+u}^{\mu\nu}\mathcal{T}_{t+u\,\mu\nu}^*}_{\text{box}-\text{like}} \Bigg], \tag{3.20}$$

where all the dependence on the gauge parameters ξ and η_μ has vanished.

We have shown that a gauge-independent decomposition is possible, and we have identified the contribution which ought to be related by the optical theorem to $\widehat{\Pi}^{\mu\nu}$. We just need to compute the other side of the optical theorem's equation: for this we just need to compute

$$\Im\mathrm{m}\,\mathcal{M}_s = \int \frac{d\Gamma}{4q^4} J^\mu(p_1, p_2) \left[\mathcal{T}_s^{\mu\nu}\mathcal{T}_{s\,\mu\nu}^* - 8\mathcal{G}\mathcal{G}^* \right] J^{*\nu}(p_1, p_2) \sin^2 \frac{k_1\vartheta k_2}{2} \sin^2 \frac{p_1\vartheta p_2}{2} =$$

$$= \int d\Gamma \frac{1}{2} J^\mu(p_1, p_2) \frac{1}{q^2} \left[4q^2 (g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) + \frac{(k_1^\mu - k_2^\mu)(k_1^\nu - k_2^\nu)}{q^4} \right] \times$$

$$\times \frac{1}{q^2} J^{*\nu}(p_1, p_2) \sin^2 \frac{k_1\vartheta k_2}{2} \sin^2 \frac{p_1\vartheta p_2}{2}. \tag{3.21}$$

We introduce ($p = k_1$ and $k_2 = q - k_1$)

$$\frac{1}{2} A^{\mu\nu} \doteq 4q^2 \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + (2p - q)^\mu (2p - q)^\nu, \tag{3.22}$$

in terms of which the optical theorem states

$$\Im\mathrm{m}\,\hat{\Pi}_{\mu\nu} = \frac{1}{2} \int \mathrm{d}\Omega \left(\frac{1 - \cos p\vartheta q}{2} \right) A_{\mu\nu}. \tag{3.23}$$

The left hand side is easily evaluated from (2.13) using

$$\Im\mathrm{m}\,K_\nu(x) = (-1)^{\nu+1} \frac{\pi}{2} J_\nu(|x|).$$

We find that for $q^2 < 0$ there are no imaginary parts, while for $q^2 > 0$

$$\Im\mathrm{m}\,\widehat{\Pi}_c(q) = \frac{q^2}{2\pi} \left(\frac{11}{12} + \frac{2\sin z - 8z^2 \sin z - 2z\cos z}{8z^3} \right)$$

$$\Im\mathrm{m}\left\{ \widehat{\Pi}_c(q) + \widehat{\Pi}_\vartheta(q) \right\} = \frac{q^2}{2\pi} \left(\frac{11}{12} + \frac{-4\sin z - 6z^2 \sin z + 4z\cos z}{8z^3} \right), \tag{3.24}$$

where $z = -\dfrac{|\tilde{q}||q|}{2}$. We consider now the right hand side, that is more easily analyzed by employing the transformations in the planes (q^0, q^1) and (q^2, q^3) admitted by the residual Lorentz invariance: without loss of generality we can take $q^\mu = (q^0, 0, 0, q^3)$ and we obtain

$$p\vartheta q = -\vartheta q^3 |\vec{p}| \cos\varphi \sin\chi = -|\tilde{q}||\vec{p}| \cos\varphi \sin\chi,$$

where (χ, φ) are polar and azimuthal angles defined with respect to x_3. The phase space integrals evaluate to

$$\frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3p\, \mathrm{d}^3k}{4E_p E_k} \frac{(1 - \cos p\vartheta q)}{2} (2\pi)^4 \delta^{(4)}(p + q + k) A_{\mu\nu} =$$
$$= \frac{1}{8(2\pi)^2} \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \sin\chi \left(\frac{1 - \cos\left(-|\tilde{q}||\vec{p}| \cos\varphi \sin\chi\right)}{2} \right) A_{\mu\nu}. \qquad (3.25)$$

We can then project $A_{\mu\nu}$ on the two independent polarizations:

$$A^{\mu\nu} \doteq I_1 \left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} - \frac{\tilde{q}_\mu \tilde{q}_\nu}{|\tilde{q}|^2} \right] + I_2 \frac{\tilde{q}_\mu \tilde{q}_\nu}{|\tilde{q}|^2}, \qquad (3.26)$$

$$I_1 = \tfrac{7}{2} q^2 - 2 \frac{(p\vartheta q)^2}{|\tilde{q}|^2} = \tfrac{7}{2} q^2 + 2 \frac{|\tilde{q}|^2 |\vec{p}|^2 \cos\varphi \sin\chi}{|\tilde{q}|^2}, \qquad (3.27)$$

$$I_2 = 4q^2 + 4 \frac{(p\vartheta q)^2}{|\tilde{q}|^2} = 4q^2 - 4 \frac{|\tilde{q}|^2 |\vec{p}|^2 \cos\varphi \sin\chi}{|\tilde{q}|^2} \qquad (3.28)$$

to obtain

$$\int \mathrm{d}\Omega \left(\frac{1 - \cos p\vartheta q}{2} \right) I_1 = \frac{q^2}{4\pi} \left(\frac{11}{12} + \frac{2\sin z - 8z^2 \sin z - 2z \cos z}{8z^3} \right),$$
$$\int \mathrm{d}\Omega \left(\frac{1 - \cos p\vartheta q}{2} \right) I_2 = \frac{q^2}{4\pi} \left(\frac{11}{12} + \frac{-4\sin z - 6z^2 \sin z + z \cos z}{8z^3} \right). \qquad (3.29)$$

This expression coincides with one half of equation (3.24). This completes our check of unitarity for the non-commutative $U(1)$ gauge theory.

3.3 Analyticity

An analytic two-point function has the property that its real and imaginary parts are related by a dispersion relation. Additional care is needed when dealing with gauge field theories off-shell, because of gauge artifacts. Unphysical gauge-dependent degrees of freedom may introduce unphysical Landau poles that, in turn, can spoil analyticity. The PT resummation prescription provides an off-shell two-point function which takes only physical Landau singularities into account. In the non-commutative setup we can use this amplitude to investigate the deformation's effect on the analytic structure of the two-point function. We start by checking the dispersion relation involving the *retarded self-energy*'s real and imaginary parts

$$\operatorname{Re}\Pi^{(\mathrm{R})}(q) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{+\infty} \mathrm{d}\omega \frac{\operatorname{Im}\Pi^{(\mathrm{R})}(\omega, \vec{q})}{\omega - q^0} = \frac{2}{\pi} \mathrm{P} \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \omega \frac{\operatorname{Im}\Pi^{(\mathrm{R})}(\omega, \vec{q})}{\omega^2 - (q^0)^2}. \qquad (3.30)$$

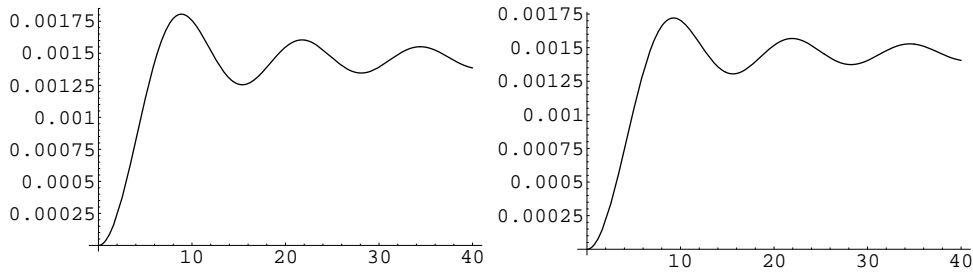**Figure 6:** The imaginary parts $\Im m\,\widehat{\Pi}_c(q)$ and $\Im m\left\{\widehat{\Pi}_c + \widehat{\Pi}_\vartheta\right\}$ as functions of $|\tilde{q}|$, for $g^2 = 10^{-2}$.

As already pointed out in [50], for positive energy, the Feynman and retarded self-energies coincide, so an expression analogous to (3.30) must hold for $\widehat{\Pi}_{\mu\nu}$. Inserting equation (3.24) into (3.30) and evaluating the integral we easily obtain the real part of the self-energy. This, in turn, agrees with the value of the real part of the self-energy extracted directly from (2.13). This shows that the two-point function's analyticity is preserved in the non-commutative setup so the resummed amplitudes propagate physically meaningful information.

The-non planar part of $\widehat{\Pi}_{\mu\nu}^g$ satisfies an *unsubtracted dispersion relation* but even though the imaginary part is regular in the $\vartheta p \to 0$ limit it gives rise, once integrated, to an IR-divergent real part. This divergence was observed in [50], and it was interpreted as an effect of the UV-IR mixing. However the off-shell amplitudes considered there are gauge-dependent, so they may contain unphysical degrees of freedom which in principle can spoil the physical nature of this divergence. Working with the pinched self-energy we proved that the IR-divergence comes out from the integration in the high energy region and so it is indeed a physical UV-IR effect.

4. The two-dimensional theory

In this section we will employ our gauge-invariant resummed self-energy to analyze the two-dimensional theory. There are several reasons to consider this apparently simple situation: first of all, bidimensional non-commutative gauge theories present a non-trivial behavior at the perturbative level, exhibiting unexpected effects when Wilson loops are evaluated. It was found in [43] that, in the large-ϑ limit, non-planar contributions are not suppressed, leading to an anomalous finite result. Later on, it was shown [44] that area-preserving diffeomorphism invariance is violated in perturbation theory, a surprising feature further confirmed by recent computations [45] (see also [46] for a nonperturbative approach to this problem).

Since all these phenomena appear at the perturbative level in gauge-invariant observables, it is quite tempting to explore the gauge-invariant propagator itself. Actually all of these results were obtained by using axial gauges, that explicitly trivialize the self-interactions of the gauge fields in two-dimensions. This procedure leads to infrared-finite results, without resorting to the choice of any explicit cut-off once a suitable prescription

for the gauge propagator is adopted. In our case, instead, we have used a covariant gauge-fixing in deriving the self-energy in D-dimensions: by taking the limit $D \to 2$ in our general expression, we will obtain automatically a dimensionally regularized result.

In both the $D \to 2$ and $\vartheta \to 0$ limits one would naively expect a vanishing gauge-invariant self-energy when matter is absent: two-dimensional gauge theories have no physical local degrees of freedom and, since we are considering the $U(1)$-case, a free theory is recovered for $\vartheta = 0$ at the classical level. This last feature should be true even in presence of matter. We anticipate that quantum effects will produce a quite different behavior instead, as we will see in the following.

4.1 The gluon contribution

We start by recalling the expression for $\widehat{\Pi}^g_{\mu\nu}$ in equation (2.12):

$$
\widehat{\Pi}^g_{\mu\nu}(q) = \frac{-g^2}{(4\pi)^{D/2}} \int_0^1 \mathrm{d}x \Bigg\{ \frac{8q^2 + (4-2D)(q^2 x^2 - M_g^2)}{(M_g^2)^{2-D/2}} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \times
$$
$$
\times \left[\Gamma(2 - D/2) - 2 \left(\frac{|\tilde{q}|M_g}{2} \right)^{2-D/2} K_{2-D/2}(M_g|\tilde{q}|) \right] +
$$
$$
+ (4-2D) \left[2 \left(\frac{1}{2M_g} \right)^{1-D/2} (|\tilde{q}|M_g)^{2-D/2} K_{-D/2}(M_g|\tilde{q}|) \right] \frac{\tilde{q}_\mu \tilde{q}_\nu}{\tilde{q}^2} \Bigg\}. \quad (4.1)
$$

We notice that for $2 < D < 4$ the limit $\vartheta \to 0$ can be taken safely in the first term, obtaining the expected decoupling in the usual transverse part, while the new transverse structure produces the well known $1/\vartheta^{(D-2)}$ divergence, generated by the IR/UV mixing. On the other hand, as $\vartheta \to \infty$ we recover the pure planar theory.

Things change when one tries to go down to $D = 2$. A first observation concerns the infrared and ultraviolet behavior. In order to understand the potential divergences and the peculiar role played by non-commutativity, it is useful to take a step back and look directly at the Feynman integral

$$
\int d^D k \left[\frac{8q^2 g_{\mu\nu}}{k^2(q+k)^2} + (4-2D) \frac{(k^2 g_{\mu\nu} - 2k_\mu k_\nu)}{k^2(q+k)^2} \right] \sin^2 \left(\frac{q \vartheta k}{2} \right). \quad (4.2)
$$

One immediately sees how non-commutativity plays a crucial role in approaching two dimensions: the potential infrared divergence in the first term is smoothed out by the $\sin^2(\frac{q\vartheta k}{2})$ term. Here non-commutativity acts as an infrared regulator when one computes the momentum integral. The second contribution is instead a classical example of an *evanescent term*, being multiplied by $(4-2D)$, that in presence of ultraviolet and infrared divergences could generate a finite result in the limit $D \to 2$.

A phenomenon of this type was noticed in [41], where Wilson loops were computed in non-abelian $D = 2$ gauge theories by using dimensional regularization. In our case non-commutativity provides a natural cut-off and the evanescent term gives no contribution in the two-dimensional limit. These features are displayed by explicitly performing the limit

$D \to 2$ in equation (4.1): we simply obtain

$$\widehat{\Pi}_{\mu\nu}^g(q) = -\frac{g^2}{(4\pi)} \left[\int_0^1 dx \frac{(8q^2)}{M^2} \left(1 - (|\tilde{q}|M_g) K_1(M_g|\tilde{q}|) \right) \right] (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}). \quad (4.3)$$

Here one can see the role played by non-commutativity: we get an infrared-regulated contribution from the first term of (4.1), which owes its finiteness to a delicate cancellation between the planar and the non-planar sectors, piloted by the non-commutative phase. We remark that the term proportional to $\tilde{q}_\mu \tilde{q}_\nu$ in equation (4.1), which displayed in $D = 4$ the known IR/UV effects, is the product of a finite term times $D - 2$ and so it vanishes.

Now we can explore a couple of different limits. Using the expansion of the Bessel function we take the limit $s = (|\tilde{q}||q|/2) \to 0$, observing that the full self-energy vanishes. We recover the free theory result, as could be well expected because no ultraviolet divergence was regularized by non-commutativity in the final expression. We will see that when matter is present the situation changes drastically. The opposite, $s \to \infty$ limit is more interesting: this computation should reproduce the planar part of a commutative non-abelian theory. In order to perform the calculation, we write down the relevant part of the self-energy as follows:

$$\int_0^1 dx \frac{q^2}{M_g^2} \left[1 - (|\tilde{q}|M_g) K_1(M_g|\tilde{q}|) \right] = 2 \int_0^1 dx \frac{1}{\sqrt{1-x}} \left[\frac{1}{x} - (|\tilde{q}||q|/2) \frac{1}{\sqrt{x}} K_1(|\tilde{q}||q|\sqrt{x}/2) \right] =$$

$$= 2 \int_0^1 dx \left[\frac{1}{\sqrt{1-x}} \frac{1}{x} - \frac{K_1(\sqrt{x})}{\sqrt{x(1-x/s^2)}} + \frac{K_1(\sqrt{x})}{\sqrt{x(1-x/s^2)}} - \frac{2s K_1(s\sqrt{x})}{\sqrt{x(1-x)}} \right]. \quad (4.4)$$

The above expression can be easily evaluated in the limit $s \to \infty$:

$$2 \left[\int_0^1 dx \left(\frac{1}{\sqrt{1-x}} \frac{1}{x} - \frac{K_1(\sqrt{x})}{\sqrt{x}} \right) + \ln(s^2) + \int_1^\infty dx \frac{K_1(\sqrt{x})}{\sqrt{x}} \right] \to 2\ln(s^2), \quad (4.5)$$

implying the following behavior of the vacuum polarization

$$\lim_{s \to \infty} \widehat{\Pi}_{\mu\nu}^g = -g^2 \frac{8}{\pi} \ln(s) \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (4.6)$$

We observe therefore a curious "twisted" incarnation of the familiar IR/UV mixing: the original infrared divergence is cured by the non-commutativity, and it reappears as an ultraviolet effect as $s \to \infty$. In this limit the effective infrared cut-off is removed, since the non-planar contribution is suppressed completely. We remark that our result is a simple example of how, in two dimensions, the limit of large-ϑ can produce non-trivial effects in perturbation theory, as shown in [43, 44] by computing Wilson loops.

4.2 Fermionic and scalar contributions

It is a simple exercise to compute the contribution to self-energy of n_f fermions and n_s scalars, taking the limit $D = 2$ in the general expressions (3.1) and (3.2):

$$\widehat{\Pi}_{\mu\nu}^f = \sum_{n_f} \frac{g^2}{\pi} \int_0^1 dx \left\{ \frac{q^2 x(1-x)}{M_f^2} \left[1 - (|\tilde{q}|M_f) K_1 \right] + (M_f|\tilde{q}|) K_1 \right\} (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}), \quad (4.7)$$

$$\widehat{\Pi}^s_{\mu\nu} = \sum_{n_s} \frac{g^2}{\pi} \int_0^1 \mathrm{d}x \left\{ \frac{q^2(x^2 - 1/4)}{M_s^2} \left[1 - (|\tilde{q}|M_s) K_1 \right] - (M_s|\tilde{q}|) K_1 \right\} (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}). \quad (4.8)$$

The above expressions are finite even in the massless case: in particular the fermionic contribution is regular independently of the cancellation between planar and non-planar sectors, and the infrared safeness of the massless scalar integral is ensured by the same mechanism as in the gauged case.

We observe, at variance with the pure gauge case, that the structure $\tilde{q}_\mu \tilde{q}_\nu / \tilde{q}^2$ leads to a finite contribution. Since in two dimensions $\vartheta_{\mu\nu} = \vartheta \varepsilon_{\mu\nu}$, it happens that $\tilde{q}_\mu \tilde{q}_\nu / \tilde{q}^2 = g_{\mu\nu} - q_\mu q_\nu / q^2$, and no new structure appears. Taking the limit $s \to 0$ we have the following surprising result:

$$\widehat{\Pi}^f_{\mu\nu} \to \sum_{n_f} \frac{g^2}{(\pi)} \int_0^1 \mathrm{d}x \, (M_f|\tilde{q}|) K_1(M_f|\tilde{q}|)(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \to n_f \frac{g^2}{\pi}(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}), \quad (4.9)$$

$$\widehat{\Pi}^s_{\mu\nu} \to -\sum_{n_s} \frac{g^2}{\pi} \int_0^1 \mathrm{d}x \, (M_f|\tilde{q}|) K_1(M_f|\tilde{q}|)(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \to -n_s \frac{g^2}{\pi}(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}). \quad (4.10)$$

We are left with a constant term which wastes the decoupling: this is a "canonical" IR/UV effect, as one can easily realize, because it originates from the $\tilde{q}_\mu \tilde{q}_\nu / \tilde{q}^2$ term. For $D > 2$ it produces the well known $1/\vartheta^{(D-2)}$ divergence, while in two dimensions it provides a finite, ϑ-independent result as the non-commutativity is sent to zero.

This effect of producing a Schwinger mass at one-loop for the gauge field is the exact analogue of the induction of a Chern-Simons term in three dimension when Majorana fermions are coupled to a non-commutative $U(1)$ theory [47]. In that case too a non-vanishing Chern-Simons term, generated by the non-commutative interaction, survives as $\vartheta \to 0$, leading to a one-loop mass for the gauge field.

The opposite situation, the limit $s \to \infty$ is straightforward to compute in the massive case, the Bessel function being exponentially suppressed, and we are left with the regular planar contributions. As we anticipated before, the massless scalar case exhibits the same anomalous behavior as the pure gauge case at large s: exploiting the same technique, we easily derive

$$\lim_{s \to \infty} \widehat{\Pi}^s_{\mu\nu} = \frac{n_s}{3} \frac{g^2}{\pi} \ln(s)(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}). \quad (4.11)$$

When $n_s = 24$ this cancels exactly the anomalous divergence coming from the gauge sector. We do not have an an explanation for this curious fine-tuning.

5. Conclusions and outlook

An interesting issue in non-commutative gauge theories is whether unitarity and analyticity of the Green functions are spoiled by non-local effects, both on-shell and off-shell. A consistent, gauge-invariant resummation formalism is required to investigate these properties, and a leading candidate is provided by the pinch techniques framework.

In this paper we have worked out a gauge-invariant resummation prescription, extending the pinch techniques to non-commutative gauge theories. We have shown how

resummed off-shell Green's functions satisfy the requirements that are usually imposed in the commutative setup to implement a consistent gauge-invariant reorganization of the perturbative expansion. In particular, an important check of the validity of the pinch techniques requires that the resummed self-energy reduce on-shell to the unpinched one. We have verified that this is the case: in particular, the pinched gluon self-energy displays, in four dimensions, the well know tachyonic divergence that leads to vacuum destabilization.

With this gauge-invariant resummation formalism at our disposal we proceeded to carry out an analysis of the optical theorem, verifying that the pinch techniques provide a powerful tool for investigating the issue of unitarity of non-commutative gauge theories. Previous analyses in this field employed techniques which were sensitive to unphysical gauge effects; our main result is a test of the optical theorem in the s-channel, employing off-shell resummed functions, for purely spacelike non-commutativity. For timelike non-commutativity we found evidence for unitarity violation both on-shell and off-shell, consistently with previously known results.

Finally we came to the analysis of the $D \to 2$ limit. The two-dimensional theory is expected to be trivial due to the absence of propagating degrees of freedom for the pure gauge sector. We found instead a non-trivial correction to the dispersion relation even in the absence of matter. Moreover, when matter is included we found an anomalous behavior in the $\vartheta \to 0$ limit. A finite term survives the commutative limit, violating the expected decoupling and inducing a mass term for the photon. In the $\vartheta \to \infty$ limit instead, a twisted version of the UV-IR mixing comes out: the original infrared divergences regulated by the non-commutativity reappear in the ultraviolet domain.

In the longer run, the most interesting issues are related with the possibility of writing down a consistent Schwinger–Dyson equation to investigate vacuum destabilization from a non-perturbative point of view. There are at least two motivations for doing so. A first reason is related to the possibility that vacuum destabilization might simply be an artifact of perturbation theory. A more speculative motivation is related to the possibility for the existence of striped phases like the ones observed in non-commutative scalar theories [24] in the gauged case. It should be remarked that a transition to a striped phase here would be particularly puzzling, since translations form a subset of the full $U(1)_\star$ gauge group. Quite surprisingly, however, we found hints that such an exotic phase might be realized in the three-dimensional topologically massive case [51, 52].

For this purpose, the next step would be to write down a gauge-invariant gap equation for the pinch-technique propagator $\widehat{\Delta}$. Truncating the Schwinger–Dyson equations to one-loop order one has

$$\hat{\Delta}^{-1}(q) = \hat{\Delta}_0^{-1}(q) + \widehat{V}^{(3)}(q,p,k)\widehat{\Delta}(p)\widehat{\Delta}(k)\widehat{V}^{(3)}(-k,-p,-q) +$$
$$+ \widehat{V}^{(4)}(q,p,-q,-p)\widehat{\Delta}(p) + \text{pinch terms}, \tag{5.1}$$

where $\widehat{V}$ are the full vertices. The pinch terms are the usual ones, but they must be computed using the exact pinch-technique propagators $\widehat{\Delta}(q)$, for which one can take an Ansatz of the form

$$\hat{\Delta}(q) = T(q)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \Theta(q)\left(\frac{\tilde{q}_\mu \tilde{q}_\nu}{q^2}\right) + (1-\xi)\left(\frac{q_\mu q_\nu}{q^4}\right). \tag{5.2}$$

The propagator's trivial dependence on the gauge fixing is retained in the framework of the pinch techniques, as explained above, and this should be used to test that the truncation is indeed self-consistent and gauge-independent.

A first trivial attempt consists, for example, in seeing what happens when one computes the gap equation (5.1) for non-commutative QCD, with all pinch terms included, using the tree-level form of the vertices. So doing one encounters encouraging cancellations among the ξ-dependent terms, but there are some gauge-fixing-dependent terms (*e.g.* those proportional to $\Theta(q)$) that don't cancel. This is not at all a surprise, since one should in principle determine the $\widehat{V}^{(3)}$ vertex through its own Scwhinger–Dyson equation. In the commutative case one can use the pinch-technique Ward identities

$$q_\alpha(\widehat{V} - \widehat{V}_0)_{\alpha\mu\nu}(q, p, -q, -p) = \widehat{\Pi}_{\mu\nu}(q + p) - \widehat{\Pi}_{\mu\nu}(p) \qquad (5.3)$$

to find the form of $\widehat{V}^{(3)}$. Hopefully an analogous approach can lead, like in the commutative case, to a meaningful, gauge-invariant gap equation.
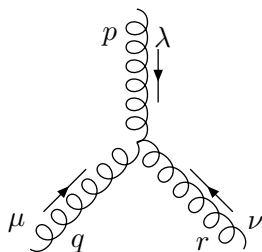
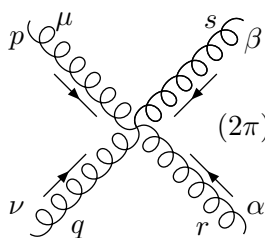## Acknowledgments

A. Feynman rules

In this appendix we shall summarize the Euclidean Feynman rules adopted in this paper

GLUON SECTOR:
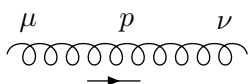
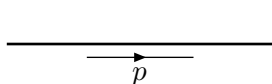$$\frac{1}{p^2}\left(g_{\mu\nu} - (1 - \xi)p_\mu p_\nu\right)$$

$$-(2\pi)^d\delta(p + q + r)2ig(g_{\lambda\mu}(p - q)_\nu + g_{\mu\nu}(q - r)_\lambda + \\ + g_{\nu\lambda}(r - p)_\mu)\sin\left(\frac{p\vartheta q}{2}\right)$$
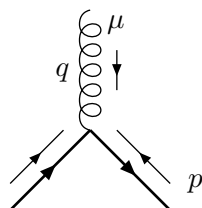

$$(2\pi)^d\delta(p + q + r + s)(-4g^2)\left(\sin\frac{p\vartheta q}{2}\sin\frac{r\vartheta s}{2}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) + \\ + \sin\frac{p\vartheta r}{2}\sin\frac{q\vartheta s}{2}(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\nu\alpha}) + \\ + \sin\frac{q\vartheta r}{2}\sin\frac{p\vartheta s}{2}(g_{\mu\nu}g_{\alpha\beta} - g_{\nu\beta}g_{\mu\alpha})\right)$$

GHOST SECTOR:

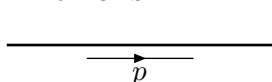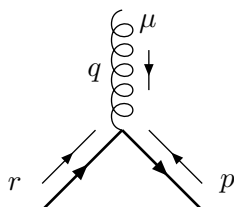

$$\frac{1}{p^2}$$

$$(2\pi)^d \delta(p+q+r) 2igp^\mu \sin\frac{p\vartheta q}{2}$$

DIRAC FERMIONS:
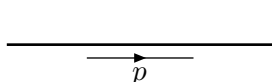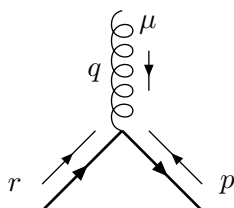

$$\frac{1}{\not{p}+m}$$

$$(2\pi)^d \delta(p+q+r) g e^{\frac{ir\vartheta p}{2}}$$

MAIORANA FERMIONS:

$$\frac{1}{\not{p}+m}$$

$$(2\pi)^d \delta(p+q+r) g \sin\left(\frac{r\vartheta p}{2}\right)$$

Where $p\vartheta q = p_\mu \vartheta^{\mu\nu} q_\nu$.

B. Relevant scalar and euclidean tensorial integrals

This appendix is devoted to the evaluation of the relevant scalar and tensorial integrals. To begin with, we shall consider the tadpole-like integral, which is taken to be identically zero in the commutative case,

$$T = \int \frac{d^d k}{(2\pi)^d} \sin^2\left(\frac{k\vartheta q}{2}\right) \frac{1}{k^2}. \tag{B.1}$$

This integral for $d \geq 2$ is ultraviolet divergent, but infrared finite. In fact the presence of the trigonometric function smooths the behavior for small momenta. This should be contrasted with the commutative counterpart where dangerous infrared divergences appear when d approaches two.

A rigorous dimensional regularization of the integral (B.1) requires its evaluation for $d < 2$ and then to define its values in $d \geq 2$ by analytic continuation. We have

$$
\begin{aligned}
T =& \frac{1}{2} \lim_{M \to 0} \int_0^\infty dt \int \frac{d^d k}{(2\pi)^d} \left(1 - e^{-ik\vartheta q}\right) e^{-t(k^2 + M^2)} = \frac{\pi^{d/2}}{2(2\pi)^d} \lim_{M \to 0} \int_0^\infty \frac{dt}{t^{d/2}} \left(1 - e^{-\frac{|\tilde{q}|^2}{4t}}\right) e^{-tM^2} = \\
=& \frac{1}{2(4\pi)^{d/2}} \lim_{M \to 0} \left(\Gamma\left(\frac{d}{2} - 1\right) (M^2)^{d/2 - 1} - 2^{\frac{d}{2}} (M^2)^{\frac{d}{2} - 1} \left(M^2 |\tilde{q}|^2\right)^{\frac{1}{2} - \frac{d}{4}} K_{d/2 - 1}(M|\tilde{q}|)\right) = \\
=& \frac{1}{2(4\pi)^{d/2}} \lim_{M \to 0} \left[\left(2^{d-4} |\tilde{q}|^{4-d} \Gamma\left(\frac{d-4}{2}\right) M^2 - 2^{d-2} |\tilde{q}|^{2-d} \Gamma\left(\frac{d}{2} - 1\right) + O(M^4)\right) + \right. \\
& \left. + M^d \left(\frac{|\tilde{q}|^2}{4} \Gamma\left(-\frac{d}{2}\right) - \frac{|\tilde{q}|^4}{32} \Gamma\left(-1 - \frac{d}{2}\right) M^2 + O(M^3)\right)\right] = -\frac{2^{d-3} |\tilde{q}|^{2-d}}{(4\pi)^{d/2}} \Gamma\left(\frac{d}{2} - 1\right). \text{ (B.2)}
\end{aligned}
$$

Around $d = 2$, we have the following expansion

$$
T = -\frac{1}{4\pi (d-2)} + \frac{\gamma}{8\pi} + \frac{\log(\pi \mu^2 |\tilde{q}|^2)}{8\pi} + O(d-2). \tag{B.3}
$$

The second scalar integral we need is given by

$$
\begin{aligned}
S =& \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k+q)^2} \sin^2\left(\frac{q\vartheta k}{2}\right) = \\
=& \frac{1}{2} \left[\int \frac{d^d k}{(2\pi)^d} \underbrace{\frac{1}{k^2 (k+q)^2}}_{(a)} - \int \frac{d^d k}{(2\pi)^d} \underbrace{\frac{1}{k^2 (k+q)^2} e^{ik\vartheta q}}_{(b)}\right]. \tag{B.4}
\end{aligned}
$$

We have

$$
S_{(a)} = \frac{1}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + x(1-x)q^2)^2} = \frac{\Gamma(2 - d/2)}{2(4\pi)^{d/2}} \int_0^1 dx (x(1-x)q^2)^{d/2 - 2}. \tag{B.5}
$$

and

$$
\begin{aligned}
S_{(b)} =& \frac{1}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\vartheta q}}{(k^2 + x(1-x)q^2)^2} = \frac{1}{2} \int_0^1 dx \int_0^\infty dt\, t \int \frac{d^d k}{(2\pi)^d} e^{ik\vartheta q - t(k^2 + x(1-x)q^2)} = \\
=& \frac{1}{2(4\pi)^{d/2}} \int_0^1 dx \int_0^\infty dt\, t^{1 - d/2} e^{-\frac{|\tilde{q}|^2}{4t} - tx(1-x)q^2} = \frac{1}{(4\pi)^{d/2}} \left(\frac{|\tilde{q}|}{2M}\right)^{2 - \frac{d}{2}} K_{\frac{d}{2} - 2}(|\tilde{q}|M), \tag{B.6}
\end{aligned}
$$

with $M = \sqrt{q^2 x (1-x)}$. Next we consider the tensorial integral

$$
I_{\mu\nu} = \int \frac{d^d k}{(2\pi)^d} \frac{(2k_\mu + q_\mu)(2k_\nu + q_\nu) - 2k^2 g_{\mu\nu}}{k^2 (k+q)^2} \sin^2\left(\frac{q\vartheta k}{2}\right). \tag{B.7}
$$

This is manifestly transverse: in fact

$$
\begin{aligned}
q^\mu I_{\mu\nu} &= \int \frac{d^d k}{(2\pi)^d} \frac{(2(k \cdot q) + q^2)(2k_\nu + q_\nu) - 2k^2 q_\nu}{k^2 (k+q)^2} \sin^2 \left(\frac{q \vartheta k}{2} \right) = \\
&= \int \frac{d^d k}{(2\pi)^d} \frac{((q+k)^2 - k^2)(2k_\nu + q_\nu) - 2k^2 q_\nu}{k^2 (k+q)^2} \sin^2 \left(\frac{q \vartheta k}{2} \right) = \\
&= \int \frac{d^d k}{(2\pi)^d} \left(\frac{(2k_\nu + q_\nu)}{k^2} - \frac{(2k_\nu + q_\nu)}{(k+q)^2} - 2 \frac{q_\nu}{(k+q)^2} \right) \sin^2 \left(\frac{q \vartheta k}{2} \right) = \\
&= \int \frac{d^d k}{(2\pi)^d} \left(\frac{q_\nu}{k^2} + \frac{q_\nu}{k^2} - 2 \frac{q_\nu}{k^2} \right) \sin^2 \left(\frac{q \vartheta k}{2} \right) = 0.
\end{aligned}
\tag{B.8}
$$

To compute this integral, first we decompose it in its planar and non planar parts

$$
\begin{aligned}
I_{\mu\nu}^{\text{plan.}} &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{(2k_\mu + q_\mu)(2k_\nu + q_\nu) - 2k^2 g_{\mu\nu}}{k^2 (k+q)^2} \\
I_{\mu\nu}^{\text{non-plan.}} &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{(2k_\mu + q_\mu)(2k_\nu + q_\nu) - 2k^2 g_{\mu\nu}}{k^2 (k+q)^2} e^{ik\vartheta q}
\end{aligned}
\tag{B.9}
$$

and then we compute them separately. To avoid the question about how to extend the different tensor structures in non integer dimensions, we shall compute the integrals following the most straightforward path, which always begins by introducing the Feynman parameters

$$
\begin{aligned}
I_{\mu\nu}^{\text{plan.}} &= \frac{1}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(2k_\mu + (1-2x)q_\mu)(2k_\nu + (1-2x)q_\nu) - 2(k-xq)^2 g_{\mu\nu}}{(k^2 + x(1-x)q^2)^2} = \\
&= \frac{1}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{2((2-d)k^2/d - x^2 q^2)g_{\mu\nu} + (1-2x)^2 q_\mu q_\nu}{(k^2 + x(1-x)q^2)^2}.
\end{aligned}
\tag{B.10}
$$

By employing the following basic result of dimensional regularization,

$$
\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^\nu}{(k^2 + M^2)^\mu} = \frac{(M^2)^{\nu-\mu+d/2}}{(4\pi)^{d/2}} \frac{\Gamma(\nu + d/2)\Gamma(\mu - \nu - d/2)}{\Gamma(d/2)\Gamma(\mu)}.
\tag{B.11}
$$

we can evaluate all the integral over momenta in eq. (B.10)

$$
\begin{aligned}
I_{\mu\nu}^{\text{plan.}} &= \frac{1}{2(4\pi)^{d/2}} \int_0^1 dx \left[\frac{2(2-d)}{d} g_{\mu\nu}(x(1-x)q^2)^{d/2-1} \frac{\Gamma(1+d/2)\Gamma(1-d/2)}{\Gamma(d/2)\Gamma(2)} + \right. \\
&\quad \left. + ((1-2x)^2 q_\mu q_\nu - 2x^2 q^2 g_{\mu\nu})(x(1-x)q^2)^{d/2-2} \frac{\Gamma(2-d/2)}{\Gamma(2)} \right] = \\
&= \frac{1}{2(4\pi)^{d/2}} (q_\mu q_\nu - q^2 g_{\mu\nu}) \int_0^1 dx \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} (1-2x)^2.
\end{aligned}
\tag{B.12}
$$

The term linear in $1 - 2x$ vanishes because it is a total derivative.

The non-planar contribution can be computed by means of the same techniques. The computation is however more tedious because of the presence of the new vector $\tilde{q}^\mu$.

$$
I_{\mu\nu}^{\text{n-pl.}} = \frac{1}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{4k_\mu k_\nu + (1-2x)^2 q_\mu q_\nu - 2(k^2 + x^2 q^2)g_{\mu\nu}}{(k^2 + x(1-x)q^2)^2} e^{ik\vartheta q} =
$$

$$=\frac{1}{2}\int_0^1 dx\int_0^\infty dt\,t\int\frac{d^dk}{(2\pi)^d}\left[4k_\mu k_\nu+(1-2x)^2 q_\mu q_\nu-2(k^2+x^2q^2)g_{\mu\nu}\right]e^{ik\vartheta q-t(k^2+x(1-x)q^2)}=$$

$$=\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int_0^1 dx\int_0^\infty dt\,t^{1-d/2}\left[4\left(\frac{g_{\mu\nu}}{2t}-\frac{\tilde q_\mu \tilde q_\nu}{4t^2}\right)+(1-2x)^2 q_\mu q_\nu-\right.$$

$$\left.-2\left(\frac{d}{2t}-\frac{|\tilde q|^2}{4t^2}+x^2q^2\right)g_{\mu\nu}\right]e^{-|\tilde q|^2/4t-tx(1-x)q^2}=$$

$$=\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int_0^1 dx\int_0^\infty dt\,t^{1-d/2}\left[g_{\mu\nu}\left(\frac{2-d}{t}+\frac{|\tilde q|^2}{2t^2}-2x^2q^2\right)\right.$$

$$\left.+(1-2x)^2 q_\mu q_\nu-\frac{\tilde q_\mu \tilde q_\nu}{t^2}\right]e^{-|\tilde q|^2/4t-tx(1-x)q^2}. \tag{B.13}$$

Now the coefficient of $g_{\mu\nu}$ can be rearranged with the help of the following identity

$$\left(\frac{|\tilde q|^2}{2t^2}-2x^2q^2\right)e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}=$$

$$=\left[2\frac{d}{dt}\left(-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2\right)+2x(1-x)q^2-2x^2q^2\right]e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}=$$

$$=\left[2\frac{d}{dt}\left(-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2\right)-(2x-1)^2q^2+(1-2x)q^2\right]e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}= \tag{B.14}$$

$$=\left[2\frac{d}{dt}\left(e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}\right)-(2x-1)^2q^2e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}-\frac{1}{t}\frac{d}{dx}\left(e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}\right)\right].$$

The last term vanishes when integrated over x since the integrand in $x=0$ and $x=1$ takes the same value. The first term instead can be rewritten as follows

$$\int_0^\infty dt\,t^{1-d/2}2\frac{d}{dt}\left(e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}\right)=-(2-d)\int_0^\infty dt\,t^{-d/2}\,e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}. \tag{B.15}$$

This contribution exactly cancels the similar contribution in eq. (B.13). Thus we are left with

$$I_{\mu\nu}^{\text{n-pl.}}=\frac{1}{2(4\pi)^{d/2}}\int_0^1 dx\int_0^\infty dt\,t^{1-d/2}\left[(q_\mu q_\nu-g_{\mu\nu}q^2)(1-2x)^2-\frac{\tilde q_\mu \tilde q_\nu}{t^2}\right]e^{-\frac{|\tilde q|^2}{4t}-tx(1-x)q^2}=$$

$$=\frac{1}{(4\pi)^{d/2}}\int_0^1 dx\left[(q_\mu q_\nu-g_{\mu\nu}q^2)(1-2x)^2\left(\frac{|\tilde q|}{2M}\right)^{2-\frac{d}{2}}K_{\frac{d}{2}-2}\left(M|\tilde q|\right)-\right.$$

$$\left.-\frac{\tilde q_\mu \tilde q_\nu}{\tilde q^2}2M^2\left(\frac{|\tilde q|}{2M}\right)^{2-\frac{d}{2}}K_{\frac{d}{2}}\left(M|\tilde q|\right)\right]. \tag{B.16}$$

References

[1] H.S. Snyder, *Quantized space-time, Phys. Rev.* **71** (1947) 38.

[2] E. Witten, *Noncommutative geometry and string field theory, Nucl. Phys.* **B 268** (1986) 253.

[3] M.R. Douglas and C.M. Hull, *D-branes and the noncommutative torus, JHEP* **02** (1998) 008 [hep-th/9711165].

[4] A. Connes, M.R. Douglas and A.S. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *JHEP* **02** (1998) 003 [hep-th/9711162].

[5] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[6] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162];
M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[7] A. Anisimov, T. Banks, M. Dine and M. Greasser, *Comments on non-commutative phenomenology*, *Phys. Rev.* **65** (2002) 085032 [hep-th/0106356].

[8] C.E. Carlson, C.D. Carone and R.F. Lebed, *Bounding noncommutative QCD*, *Phys. Lett.* **B 518** (2001) 201 [hep-ph/0107291].

[9] C.E. Carlson, C.D. Carone and R.F. Lebed, *Supersymmetric noncommutative QED and Lorentz violation*, *Phys. Lett.* **B 549** (2002) 337 [hep-ph/0209077].

[10] For a review see J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, hep-th/0102076 and references therein .

[11] A.S. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034];
L. Alvarez-Gaumé and J.L.F. Barbon, *Morita duality and large-N limits*, *Nucl. Phys.* **B 623** (2002) 165 [hep-th/0109176].

[12] L. Alvarez-Gaumé and M.A. Vazquez-Mozo, *General properties of noncommutative field theories*, *Nucl. Phys.* **B 668** (2003) 293 [hep-th/0305093].

[13] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020 [hep-th/9912072].

[14] V.V. Khoze and G. Travaglini, *Wilsonian effective actions and the IR/UV mixing in noncommutative gauge theories*, *JHEP* **01** (2001) 026 [hep-th/0011218].

[15] L. Griguolo and M. Pietroni, *Wilsonian renormalization group and the non-commutative IR/UV connection*, *JHEP* **05** (2001) 032 [hep-th/0104217].

[16] H. Grosse and R. Wulkenhaar, *Renormalisation of scalar quantum field theory on noncommutative* $\mathbb{R}^4$, *Fortschr. Phys.* **53** (2005) 634.

[17] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, *JHEP* **12** (2000) 002 [hep-th/0002075].

[18] A. Armoni and E. Lopez, *UV/IR mixing via closed strings and tachyonic instabilities*, *Nucl. Phys.* **B 632** (2002) 240 [hep-th/0110113];
A. Armoni, E. Lopez and A.M. Uranga, *Closed strings tachyons and non-commutative instabilities*, *JHEP* **02** (2003) 020 [hep-th/0301099].

[19] J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, *Nonperturbative dynamics of noncommutative gauge theory*, *Phys. Lett.* **B 480** (2000) 399 [hep-th/0002158]; *Finite N matrix models of noncommutative gauge theory*, *JHEP* **11** (1999) 029 [hep-th/9911041].

[20] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model*, *Nucl. Phys.* **B 679** (2004) 66 [hep-th/0307075].

[21] H. Grosse and H. Steinacker, *Finite gauge theory on fuzzy* $\mathbb{CP}^2$, *Nucl. Phys.* **B 707** (2005) 145 [hep-th/0407089].

[22] W. Behr, F. Meyer and H. Steinacker, *Gauge theory on fuzzy* $S^2 \times S^2$ *and regularization on noncommutative* $\mathbb{R}^4$, *JHEP* **07** (2005) 040 [hep-th/0503041].

[23] J.M. Cornwall, R. Jackiw and E. Tomboulis, *Effective action for composite operators*, *Phys. Rev.* **D 10** (1974) 2428.

[24] S.S. Gubser and S.L. Sondhi, *Phase structure of non-commutative scalar field theories*, *Nucl. Phys.* **B 605** (2001) 395 [hep-th/0006119].

[25] G. Mandanici, *Cornwall-Jackiw-Tomboulis effective potential for canonical noncommutative field theories*, *Int. J. Mod. Phys.* **A 19** (2004) 3541 [hep-th/0304090].

[26] P. Castorina and D. Zappala, *Nonuniform symmetry breaking in noncommutative* $\lambda\varphi^{\star 4}$ *theory*, *Phys. Rev.* **D 68** (2003) 065008 [hep-th/0303030].

[27] J. Ambjørn and S. Catterall, *Stripes from (noncommutative) stars*, *Phys. Lett.* **B 549** (2002) 253 [hep-lat/0209106];
W. Bietenholz, F. Hofheinz and J. Nishimura, *The continuum limit of the non-commutative* $\lambda\varphi^{\star 4}$ *model*, hep-th/0407012.

[28] J.M. Cornwall, in Proceedings of the French-American seminar on *Theoretical aspects of QCD*, Marseille, France, 1981, J.W. Dash ed., Centre de Physique, CPT-81/P-1345.

[29] J.M. Cornwall, *Dynamical mass generation in continuum QCD*, *Phys. Rev.* **D 26** (1982) 1453;
J.M. Cornwall, W.-S. Hou and J.E. King, *Gauge invariant calculations in finite temperature QCD: landau ghost and magnetic mass*, *Phys. Lett.* **B 153** (1985) 173.

[30] J. Haeri, Bijan, *The ultraviolet improved gauge technique and the effective quark propagator in QCD*, *Phys. Rev.* **D 38** (1988) 3799;
J.M. Cornwall and J. Papavassiliou, *Gauge invariant three gluon vertex in QCD*, *Phys. Rev.* **D 40** (1989) 3474;
J. Papavassiliou, *Gauge invariant proper selfenergies and vertices in gauge theories with broken symmetry*, *Phys. Rev.* **D 41** (1990) 3179; *The gauge invariant four gluon vertex and its ward identity*, *Phys. Rev.* **D 47** (1993) 4728;
G. Degrassi and A. Sirlin, *Gauge invariant selfenergies and vertex parts of the standard model in the pinch technique framework*, *Phys. Rev.* **D 46** (1992) 3104.

[31] J. Papavassiliou and A. Pilaftsis, *A gauge independent approach to resonant transition amplitudes*, *Phys. Rev.* **D 53** (1996) 2128 [hep-ph/9507246].

[32] J. Papavassiliou and A. Pilaftsis, *Gauge-invariant resummation formalism for two point correlation functions*, *Phys. Rev.* **D 54** (1996) 5315 [hep-ph/9605385].

[33] J.M. Cornwall, W.S. Hou and J.E. King, *Gauge invariant calculations in finite temperature QCD: landau ghost and magnetic mass*, *Phys. Lett.* **B 153** (1985) 173;
J.M. Cornwall, *On one-loop gap equations for the magnetic mass in* $D = 3$ *gauge theory*, *Phys. Rev.* **D 57** (1998) 3694 [hep-th/9710128];
K. Sasaki, *Gauge-independent thermal* β *function in Yang-Mills theory via the pinch technique*, *Nucl. Phys.* **B 472** (1996) 271 [hep-ph/9512434].

[34] K. Philippides and A. Sirlin, *Consistency condition for the pinch technique self-energies at two loops*, *Nucl. Phys.* **B 477** (1996) 59 [hep-ph/9602404];

N.J. Watson, *The pinch technique beyond one loop: the gauge-independent two-loop quark self-energy*, *Nucl. Phys.* **B 552** (1999) 461 [`hep-ph/9812202`];
D. Binosi and J. Papavassiliou, *Pinch technique self-energies and vertices to all orders in perturbation theory*, *J. Phys.* **G 30** (2004) 203 [`hep-ph/0301096`].

[35] J. Gomis and T. Mehen, *Space-time noncommutative field theories and unitarity*, *Nucl. Phys.* **B 591** (2000) 265 [`hep-th/0005129`].

[36] L. Alvarez-Gaumé, J.L.F. Barbon and R. Zwicky, *Remarks on time-space noncommutative field theories*, *JHEP* **05** (2001) 057 [`hep-th/0103069`].

[37] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, *On the unitarity problem in space/time noncommutative theories*, *Phys. Lett.* **B 533** (2002) 178 [`hep-th/0201222`];
Ultraviolet finite quantum field theory on quantum spacetime, *Commun. Math. Phys.* **237** (2003) 221 [`hep-th/0301100`].

[38] Y. Liao and K. Sibold, *Time-ordered perturbation theory on noncommutative spacetime, II. Unitarity*, *Eur. Phys. J.* **C 25** (2002) 479 [`hep-th/0206011`].

[39] A. Bassetto, L. Griguolo, G. Nardelli and F. Vian, *On the unitarity of quantum gauge theories on noncommutative spaces*, *JHEP* **07** (2001) 008 [`hep-th/0105257`].

[40] F.T. Brandt, A. Das and J. Frenkel, *General structure of the photon self-energy in non-commutative QED*, *Phys. Rev.* **D 65** (2002) 085017 [`hep-th/0112127`].

[41] A. Bassetto, F. De Biasio and L. Griguolo, *Lightlike Wilson loops and gauge invariance of Yang-Mills theory in (1+1)-dimensions*, *Phys. Rev. Lett.* **72** (1994) 3141 [`hep-th/9402029`].

[42] A. Bassetto and L. Griguolo, *Two-dimensional QCD, instanton contributions and the perturbative Wu-Mandelstam-Leibbrandt prescription*, *Phys. Lett.* **B 443** (1998) 325 [`hep-th/9806037`].

[43] A. Bassetto, G. Nardelli and A. Torrielli, *Perturbative Wilson loop in two-dimensional non-commutative Yang-Mills theory*, *Nucl. Phys.* **B 617** (2001) 308 [`hep-th/0107147`];
A. Bassetto, G. Nardelli and A. Torrielli, *Scaling properties of the perturbative Wilson loop in two-dimensional non-commutative Yang-Mills theory*, *Phys. Rev.* **D 66** (2002) 085012 [`hep-th/0205210`].

[44] J. Ambjørn, A. Dubin and Y. Makeenko, *Wilson loops in 2D noncommutative euclidean gauge theory, I. Perturbative expansion*, *JHEP* **07** (2004) 044 [`hep-th/0406187`].

[45] A. Bassetto, G. De Pol, A. Torrielli and F. Vian, *On the invariance under area preserving diffeomorphisms of noncommutative Yang-Mills theory in two dimensions*, *JHEP* **05** (2005) 061 [`hep-th/0503175`].

[46] M. Cirafici, L. Griguolo, D. Seminara and R.J. Szabo, *Morita duality and noncommutative Wilson loops in two dimensions*, *JHEP* **10** (2005) 030 [`hep-th/0506016`].

[47] C.-S. Chu, *Induced Chern-Simons and WZW action in noncommutative spacetime*, *Nucl. Phys.* **B 580** (2000) 352 [`hep-th/0003007`].

[48] A. Pilaftsis, *Generalized pinch technique and the background field method in general gauges*, *Nucl. Phys.* **B 487** (1997) 467 [`hep-ph/9607451`];
D. Binosi and J. Papavassiliou, *Pinch technique self-energies and vertices to all orders in perturbation theory*, *J. Phys.* **G 30** (2004) 203 [`hep-ph/0301096`].

[49] N.J. Watson, *Universality of the pinch technique gauge boson selfenergies*, *Phys. Lett.* **B 349** (1995) 155 [hep-ph/9412319].

[50] F.T. Brandt, A. Das and J. Frenkel, *Dispersion relations for the self-energy in non-commutative field theories*, *Phys. Rev.* **D 66** (2002) 065017 [hep-th/0206058].

[51] N. Caporaso, L. Griguolo, S. Pasquetti and D. Seminara, *Non-commutative topologically massive gauge theory*, hep-th/0409109.

[52] N. Caporaso, L. Griguolo, S. Pasquetti and D. Seminara, *in preparation.*